Derivation of Complex Isolated IIR Pole

Donald Daniel

August 15, 2008

Abstract

The complex isolated IIR pole is derived. References given derive a zero from the pole, and illustrate how they are applied to make complex time domain filters.

We will start by deriving an IIR representation for a negative real pole, then generalize to the complex case. A formal derivation might derive the difference equation from a differential equation, but a heuristic derivation is more accessible, and will be presented here.

We want a unity gain negative real pole, a "leaky integrator". A good guess is:

$$y_k = (1-\epsilon)y_{k-1} + \epsilon x_k. \tag{1}$$

It is not obvious from the right side of the equation that the gain is unity, because of the y_{k-1} term. If, however, we assume that the gain is unity, and we set the x_k input level constant, then, $x_k = x_{k-1}$, $y_k = y_{k-1}$, and $y_k = x_k$. Then the equation becomes

$$y_k = (1-\epsilon)x_k + \epsilon x_k. \tag{2}$$

Since $(1-\epsilon)+\epsilon = 1$, our original equation is consistent with our assumption that the gain is unity. If we accept this, then we must look at decay rate and the complex case. Up to this point x_k and y_k could be real numbers, but they will have to be complex for the next case where we move up the frequency axis.

We assume the samples are evenly spaced in time with spacing T. At radian frequency ω, $x_k = e^{j\omega T}x_{k-1}$. If we are able to make an analogous equation with unity gain at ω_p, then $y_k = e^{j\omega_p T} y_{k-1}$. If we change our equation to

$$y_k = (1-\epsilon)e^{j\omega_p T} y_{k-1} + \epsilon x_k \tag{3}$$

then the logic of equation (1) implies that this equation has unity gain at ω_p.

If $|x_k| = 1$ for a long time, $|y_k|$ will be 1 also. If after a certain time $x_k = 0$, then y_k will decay exponentially.

The decay of a negative real pole is given in terms of continuous time t by $e^{\sigma t}$ where σ is a negative real number. The decay in each time interval will be multiplied by the amplitude at the start of the interval to get the decay at the end of the interval. Since in general $a^n a^m = a^{n+m}$, in discreet time T the decay would be $e^{\sigma k T}$ after k time intervals. If we replace $(1-\epsilon)$ with $e^{\sigma k T}$ then, since $(1-\epsilon)+\epsilon = 1$, we must replace ϵ with $(1-e^{\sigma T})$.

In the analog world T is reserved for temperature and t for continuous time. While δt could be used for discreet time, I have chosen to use t for discreet time. With this choice, the final form of the IIR pole becomes:

$$y_k = e^{\sigma_p t} e^{j\omega_p t} y_{k-1} + (1-e^{\sigma_p t}) x_k. \tag{4}$$

This equation has unity gain at ω_p. This equation can be used to derive an FIR zero, as shown in [1]. The technique of using the isolated complex poles and zeroes to make general complex filters is described in [2]. Example filters are plotted in [3], which has a link to the details of the computations.

References

[1] D.Daniel, "The Relation between IIR pole and FIR Zero", www.waltzballs.org/ other/engr/fltr3.pdf, Feb 2008.

[2] D.Daniel, "Complex Digital Filters Using Isolated Poles and Zeroes" www.waltzballs.org/ other/engr/fltr2.pdf, Jan 2008.

[3] D.Daniel, "Example Digital Filters" www.waltzballs.org/ other/ engr/ pzex.html, Jan 2008.