The Relation between IIR pole and FIR Zero

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Abstract

The reciprocal relation between the IIR pole and the FIR zero is shown.

In a previous paper [1], an isolated IIR pole and an isolated FIR zero were presented with their associated gain functions. The gain of the pole was chosen in an unusual way to make it easy to plot on the same grid used to plot filters. Here we normalize the pole gain the same as the zero gain to clarify the reciprocal relation between them.

We now repeat the pole and its gain function as given in the previous paper.

Let t be the time step, a constant. The sample number is k, the input stream is x_k, the output stream is y_k.

A pole at (σ_p, ω_p) radians/sec is given by:

$$y_k = e^{\sigma_p t} e^{j\omega_p t} y_{k-1} + (1 - e^{\sigma_p t}) x_k \tag{1}$$

Arbitrary normalization has been chosen to make the gain unity at the nearest point on the frequency circle.

In the steady state $y_k = e^{j\omega t} y_{k-1}$. Rearranging, $y_{k-1} = y_k e^{-j\omega t}$. With this substitution, solving for the steady state gain of the pole as a function of frequency ω we have:

$$y_k/x_k = (1 - e^{\sigma_p t})/(1 - e^{\sigma_p t} e^{j(\omega_p - \omega)t}) \tag{2}$$

The Nyquist rule requires two samples per cycle at the highest frequency f_m represented in a time domain digital simulation. Thus the sample period t is given by $2t = 1/f_m$, and $\omega_m t = \pi$. If a pole is at ω_p, the opposite side of the frequency circle is a distance of ω_m in either direction around the circumference of the circle.

In the gain equation if $(\omega_p - \omega) = \omega_m$ then $e^{j(\omega_p - \omega)t} = -1$ by Euler's formula, $e^{j\theta} = cos(\theta) + jsin(\theta)$. Therefore on the opposite side of the frequency circle from the pole the gain is $y_k/x_k = (1 - e^{\sigma_p t})/(1 + e^{\sigma_p t})$. The gain term in equation (1) is the coefficient of x_k. If we divide this gain term by the gain on the opposite side of the frequency circle, we will normalize the pole to have unity gain on the opposite side of the frequency circle the way the zero was normalized. The result is

$$y_k = e^{\sigma_p t} e^{j\omega_p t} y_{k-1} + (1 + e^{\sigma_p t}) x_k \tag{3}$$

The corresponding gain formula is:

$$y_k/x_k = (1 + e^{\sigma_p t})/(1 - e^{\sigma_p t} e^{j(\omega_p - \omega)t}) \tag{4}$$

We now repeat the zero from the previous paper. A zero at (σ_z, ω_z) radians/sec is given by:

$$y_k = (x_k - e^{\sigma_z t} e^{j\omega_z t} x_{k-1})/(1 + e^{\sigma_z t}) \tag{5}$$

Arbitrary normalization has been chosen to make the gain unity at the farthest point on the frequency circle.

In the steady state $x_k = e^{j\omega t} x_{k-1}$. Rearranging $x_{k-1} = x_k e^{-j\omega t}$. With this substitution, solving for the steady state gain of the zero as a function of radian frequency ω we have:

$$y_k/x_k = (1 - e^{\sigma_z t} e^{j(\omega_z - w)t})/(1 + e^{\sigma_z t}) \tag{6}$$

We now have both a pole and a zero normalized to unity on the opposite side of the frequency circle. But one is IIR and one is FIR. If we take the reciprocal of the pole, we should get a zero.

It is not clear that we could take the reciprocal of the pole and zero directly since they involve terms in both k and $k - 1$, but we can certainly take the recproval of the gain terms and work backward. But the pole gain and the zero gain formulas are already reciprocals. The reciprocal of the IIR pole is the FIR zero. There is no way to work backward from the zero gain formula to an IIR zero.

References

[1] D. Daniel, "Complex Digital Filters Using Isolated Poles and Zeroes" www.waltzballs.org/ other/engr/fltr2.pdf, Jan 2008.