Complex Digital Filters Using Isolated Poles and Zeroes

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Abstract

Convenient normalization is given for digital filters compounded from isolated poles and zeroes. Application to computationally efficient narrowband simulations is discussed.

Digital filters commonly accept a stream of real numbers as input and produce a stream of real numbers as output. A single filter can have many poles and zeroes. An alternative approach constructs a complicated filter as elemental filters in series, each representing an isolated pole or zero. Each elemental filter operates on a stream of complex numbers. While real signals can be represented as complex signals with zero imaginary part, there are applications which require complex signals with non-zero imaginary parts [1].

Let t be the time step, a constant. The sample number is k, the input stream is x_k, the output stream is y_k.

A pole at (σ_p, ω_p) radians/sec is given by:

$$y_k = e^{\sigma_p t} e^{j\omega_p t} y_{k-1} + (1 - e^{\sigma_p t}) x_k \tag{1}$$

Arbitrary normalization has been chosen to make the gain unity at the nearest point on the frequency circle.

In the steady state $y_k = e^{j\omega t} y_{k-1}$. Rearranging, $y_{k-1} = y_k e^{-j\omega t}$. With this substitution, solving for the steady state gain of the pole as a function of frequency ω we have:

$$y_k/x_k = (1 - e^{\sigma_p t})/(1 - e^{\sigma_p t} e^{j(\omega_p - \omega)t}) \tag{2}$$

A zero at (σ_z, ω_z) radians/sec is given by:

$$y_k = (x_k - e^{\sigma_z t} e^{j\omega_z t} x_{k-1})/(1 + e^{\sigma_z t}) \tag{3}$$

Arbitrary normalization has been chosen to make the gain unity at the farthest point on the frequency circle.

In the steady state $x_k = e^{j\omega t} x_{k-1}$. Rearranging $x_{k-1} = x_k e^{-j\omega t}$. With this substitution, solving for the steady state gain of the zero as a function of radian frequency ω we have:

$$y_k/x_k = (1 - e^{\sigma_z t} e^{j(\omega_z - w)t})/(1 + e^{\sigma_z t}) \tag{4}$$

The poles and zeroes are connected in series to create a filter. We choose a radian frequency ω_1 where we wish the filter gain to be unity. We take the reciprocal of the gains of the poles and zeroes at ω_1. The product of these reciprocals is the gain term for the filter. If we were to lump the gain terms at one point in the calculation, we would risk numerical difficulties. It is perhaps safer to apply each reciprocal gain term to each pole and zero respectively.

We can test the frequency response of the filter by feeding it a one, $(1, 0)$, followed by zeroes, then applying a fast Fourier transform (FFT) to the resulting output. The left and right halves of the output of the FFT will have to be swapped to put zero frequency in the middle, with negative frequencies on the left and positive frequencies on the right.

We will want to compare the FFT plot of our filter with the ideal transfer function. For zeroes z and poles p the transfer function can be factored into terms $(s - z)$ in the numerator and $(s - p)$ in the denominator, to be evaluated along the $s = j\omega$ axis. The transfer function must be normalized in the same way as the filter to compare them. Replace each zero factor in the numerator by $(j\omega - z)/(j\omega_1 - z)$, and each pole factor in the denominator by $(j\omega - p)/(j\omega_1 - p)$.

Now, an application for which this technique is well suited. Consider the case of modeling an analog bandpass filter whose bandwidth is a small fraction of its center frequency. Suppose the signal to be simulated is a carrier in the passband modulated within a narrow bandwidth. This real signal can be represented as the sum of positive frequency and negative frequency components, each complex. We use only the positive frequency component of the signal in the simulation, and only poles and zeroes with a positive imaginary coordinate. Simulating only the positive frequency poles and zeroes produces only an approximation to the correct filtering in the absence of poles and zeroes at conjugate positions, and at the origin. Out of the passband down on the skirts the approximation gets progressively worse. The carrier frequency can be displaced to zero while still preserving the modulation. We can displace the carrier this way with a corresponding displacement of the imaginary coordinate of the poles and zeroes to effect a large reduction in simulation bandwidth and computational load, in some cases justifying the approximation.

For example filters with a link to all the details of the calculations see [2]. For the derivation of the pole see [3]. For the dereviation of the zero from the pole see [4].

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References

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