Derivation of Complex Isolated IIR Pole

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Abstract

The complex isolated IIR pole is derived. References given derive a zero from the pole, and illustrate how they are applied to make complex time domain filters.

We will start by deriving an IIR representation for a negative real pole, then generalize to the complex case. A formal derivation might derive the difference equation from a differential equation, but a heuristic derivation is more accessible, and will be presented here.

We want a unity gain negative real pole, a "leaky integrator". Let ϵ be a positive real number much smaller than one. A good guess is:

$$y_k = (1-\epsilon)y_{k-1} + \epsilon x_k \tag{1}$$

It is not obvious from the right side of the equation that the gain is unity, because of the y_{k-1} term. The steady state is defined by the condition where $x_k = x_{k-1}$ for many iterations. The filter does not need to operate in the steady state, but the steady state is useful for discovering the properties of the filter. In the steady state $y_k = y_{k-1}$. If we assume the gain is unity, then $x_k = y_k$. Then the equation becomes

$$y_k = (1-\epsilon)x_k + \epsilon x_k \tag{2}$$

Since $(1-\epsilon)+\epsilon = 1$, our original equation is consistent with our assumption that the gain is unity. If we accept this, then we must look at decay rate and the complex case.

If $x_k = 1$ for a long time, then $y_k = 1$ also. If after being constant a long time $x_k = 0$ for the rest of time, then y_k will decay exponentially. This is because with x_k zero, the equation becomes:

$$y_k = (1 - \epsilon)y_{k-1} \tag{3}$$

Each y_k is a fraction less than one of the previous y_k. The fraction is the same each time. We recognize this as exponential decay. Furthermore, the amount of decay in one sample interval T is the multiplicative factor $(1 - \epsilon)$.

The decay of a negative real pole in an analog filter is given in terms of continuous time t by $e^{\sigma_p t}$ where σ_p is a negative real coordinate of a pole on the complex plane. If the continuous time interval t was equal to the discrete time interval T, then the decay during a discrete time interval would be $e^{\sigma_p T}$. Since our filter exponentially decays by a factor of $(1 - \epsilon)$ in time T, with the proper choice of ϵ we have $(1 - \epsilon) = e^{\sigma_p T}$. If we replace $(1 - \epsilon)$ with $e^{\sigma_p T}$ then, since $(1 - \epsilon) + \epsilon = 1$, we must replace ϵ with $(1 - e^{\sigma_p T)}$.

Up to this point x_k and y_k could be real numbers, but they will have to be complex for the next case where we move up the frequency axis.

We assume the samples are evenly spaced in time with spacing T. At radian frequency ω, in the steady state $x_k = e^{j\omega T}x_{k-1}$. If we are able to make an analogous equation with unity gain at ω_p, then in the steady state $y_k = e^{j\omega_p T}y_{k-1}$. If we change our equation to

$$y_k = e^{\sigma_p T}e^{j\omega_p T}y_{k-1} + (1 - e^{\sigma_p T})x_k \tag{4}$$

then the logic of equation (1) implies that this equation has unity gain at ω_p. More importantly for its validity as a pole, in the absence of an input signal it decays exponentially at its natural frequency.

In the analog world T is reserved for temperature to compute thermal noise in high gain circuits, and t for continuous time. While δt could be used for discrete time, I have chosen to use t for discrete time. With this choice, the final form of the IIR pole becomes:

$$y_k = e^{\sigma_p t}e^{j\omega_p t}y_{k-1} + (1 - e^{\sigma_p t})x_k \tag{5}$$

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This equation has unity gain at ω_p. This equation can be used to derive an FIR zero, as shown in [1]. The technique of using the isolated complex poles and zeroes to make general complex filters is described in [2]. Example filters are plotted in [3], which has a link to the details of the computations.

References

[1] D.Daniel, "The Relation Between IIR Pole and FIR Zero". www.waltzballs.org/ other/ engr/ fltr3.pdf, Feb, 1008.

[2] D. Daniel, "Complex Digital Filters Using Isolated Poles and Zeroes" www.waltzballs.org/ other/ engr/ fltr2.pdf, Jan 2008.

[3] D. Daniel, "Example Digital Filters" www.waltzballs.org/ other/ engr/ pzex.html, Jan 2008.