The Relation Between IIR Pole and FIR Zero

Donald Daniel

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Abstract

A digital filter representing an isolated zero is derived from a digital filter representing an isolated pole. Even though the pole is IIR, the zero is unavoidably FIR.

In a previous paper [1], an isolated IIR pole and an isolated FIR zero were presented with their associated gain functions. The pole was derived in [2]. We now derive the zero from the pole.

We now repeat the pole and its gain function as given in the previous papers.

Let t be the time step, a constant. The sample number is k, the input stream is x_k, the output stream is y_k.

A pole at (σ_p, ω_p) radians/sec is given by:

$$y_k = e^{\sigma_p t} e^{j\omega_p t} y_{k-1} + (1 - e^{\sigma_p t}) x_k \tag{1}$$

The pole is a circular function of ω. If the ω coordinate defines a location on a frequency circle, zero frequency is on one side of the circle, and the positive and negative Nyquist frequencies are at the same point on the opposite side of the frequency circle. If σ_p is kept the same, but ω_p is changed, the shape of the function evaluated along the frequency circle does not change, it just rotates around the circle to follow the change in ω_p. The normalization used makes the gain unity at ω_p. Moving away from ω_p on the frequency circle, the function decreases to a minimum value on the opposite side of the frequency circle.

In the steady state $y_k = e^{j\omega t} y_{k-1}$. Rearranging, $y_{k-1} = y_k e^{-j\omega t}$. With this substitution, solving for the steady state gain of the pole as a function of frequency ω we have:

$$y_k/x_k = (1 - e^{\sigma_p t})/(1 - e^{\sigma_p t} e^{j(\omega_p - \omega)t}) \quad (2)$$

If s is a complex number that could be anywhere in the complex plane, then in the transfer function for a filter, a zero at z is represented by $(s - z)$, and a pole at p is represented by $1/(s - p)$. A pole and a zero at the same location on the complex plane are reciprocals of each other. To derive a zero from the pole we already have, we will take the reciprocal of the gain function and work backward to the equation for the zero.

The normalizations chosen for the final version of the pole, and for the final version of the zero, are chosen to make it easy to plot them with the same plotting program used to plot the frequency response of completed filters. To normalize the zero the way we want, first, we re-normalize the pole so that it is unity gain on the opposite side of the frequency circle from ω_p.

The Nyquist rule requires two samples per cycle at the highest frequency f_m represented in a time domain digital simulation. Thus the sample period t is given by $2t = 1/f_m$, and $\omega_m t = \pi$. If a pole is at ω_p, the opposite side of the frequency circle is a distance of ω_m in either direction around the circumference of the circle.

In the gain equation if $(\omega_p - \omega) = \omega_m$ then $e^{j(\omega_p - \omega)t} = -1$ by Euler's formula, $e^{j\theta} = cos(\theta) + jsin(\theta)$. Therefore on the opposite side of the frequency circle from the pole the gain is $y_k/x_k = (1 - e^{\sigma_p t})/(1 + e^{\sigma_p t})$. The gain term in equation (1) is the coefficient of x_k. If we divide this gain term by the gain on the opposite side of the frequency circle, we will normalize the pole to have unity gain on the opposite side of the frequency circle. The result is

$$y_k = e^{\sigma_p t} e^{j\omega_p t} y_{k-1} + (1 + e^{\sigma_p t}) x_k \quad (3)$$

The corresponding gain formula is:

$$y_k/x_k = (1 + e^{\sigma_p t})/(1 - e^{\sigma_p t} e^{j(\omega_p - \omega)t}) \quad (4)$$

Our zero should have a gain function that is the reciprocal of this, or

$$y_k/x_k = (1 - e^{\sigma_z t} e^{j(\omega_z - \omega)t})/(1 + e^{\sigma_z t}) \quad (5)$$

To make the next step clearer we factor a bit:

$$y_k/x_k = (1 - e^{\sigma_z t} e^{j\omega_z t} e^{-j\omega t})/(1 + e^{\sigma_z t}) \tag{6}$$

In the steady state there are two conditions that we might use to work backward to the equation for the zero: $y_{k-1} = y_k e^{-j\omega t}$ and $x_{k-1} = x_k e^{-j\omega t}$.

If we solve for y_k we will get a term of $x_k e^{-j\omega t}$. Making the appropriate substitution, we have the equation for the zero:

$$y_k = (x_k - e^{\sigma_z t} e^{j\omega_z t} x_{k-1})/(1 + e^{\sigma_z t}) \tag{7}$$

Note that the pole was IIR, but the zero derived from it is FIR.

References

[1] D. Daniel, "Complex Digital Filters Using Isolated Poles and Zeroes" www.waltzballs.org/ other/engr/fltr2.pdf, Jan 2008.

[2] D. Daniel, "Derivation of Complex Isolated IIR Pole" www.waltzballs.org/other/engr/fltr4.pdf, Aug 15, 2008, revised Apr 12, 2011.